FUNCTIONS ON SURFACES AND INCOMPRESSIBLE SUBSURFACES

SERGIY MAKSYMENKO

Abstract. Let M be a smooth connected compact surface, P be either a real line $\mathbb{R}$ or a circle S^1. Then we have a natural *right* action of the group $\mathcal{D}(M)$ of diffeomorphisms of M on $\mathcal{C}^\infty(M,P)$. For $f \in \mathcal{C}^\infty(M,P)$ denote respectively by $\mathcal{S}(f)$ and $\mathcal{O}(f)$ its stabilizer and orbit with respect to this action. Recently, for a large class of smooth maps $f : M \to P$ the author calculated the homotopy types of the connected components of $\mathcal{S}(f)$ and $\mathcal{O}(f)$. It turned out that except for few cases the identity component of $\mathcal{S}(f)$ is contractible, $\pi_i\mathcal{O}(f) = \pi_i M$ for $i \geq 3$, and $\pi_2\mathcal{O}(f) = 0$, while $\pi_1\mathcal{O}(f)$ it only proved to be a finite extension of $\pi_1\mathcal{D}_{\mathrm{id}}(M) \oplus \mathbb{Z}^l$ for some $l \geq 0$. In this note it is shown that if $\chi(M) < 0$, then $\pi_1\mathcal{O}(f) = G_1 \times \cdots \times G_n$, where each G_i is a fundamental group of the restriction of f to a subsurface $B_i \subset M$ being either a 2-disk or a cylinder or a Möbius band. For the proof of main result incompressible subsurfaces and cellular automorphisms of surfaces are studied.

1. Introduction

Let M be a smooth compact connected surface and P be either the real line $\mathbb{R}$ or the circle S^1. Consider the *right* action of the group $\mathcal{D}(M)$ of diffeomorphisms of M on $\mathcal{C}^\infty(M,P)$ defined by

$$h \cdot f = f \circ h^{-1}$$

for $h \in \mathcal{D}(M)$ and $f \in \mathcal{C}^\infty(M,P)$. For every $f \in \mathcal{C}^\infty(M,P)$ let

$$\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\},$$

$$\mathcal{S}(f) = \{h \mid f = f \circ h, \ h \in \mathcal{D}(M)\}$$

be respectively the orbit and the stabilizer of f with respect to this action. We will endow $\mathcal{D}(M)$, $\mathcal{S}(f)$, $\mathcal{C}^\infty(M,P)$, and $\mathcal{O}(f)$ with the

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corresponding topologies $\mathcal{C}^\infty$. Denote by $\mathcal{S}_{\mathrm{id}}(f)$ the identity path component of $\mathcal{S}(f)$ and by $\mathcal{O}_f(f)$ the path component of f in $\mathcal{O}(f)$. In [10] the author calculated the homotopy types of $\mathcal{S}_{\mathrm{id}}(f)$ and $\mathcal{O}_f(f)$ for all Morse maps $f : M \to P$.

Moreover, in [12] the results of [10] were extended to a large class of maps with (even degenerate) isolated critical points satisfying certain "non-degeneracy" conditions. In fact there were introduced three types of isolated critical points (called S, P, and N) and the following three axioms for f:

- (Bd) *f takes constant value at each connected component of ∂M and $\Sigma_f \subset \mathrm{Int} M$.*
- (SPN) *Every critical point of f is either an S- or a P- or an N-point.*
- (Fibr) *The natural map $p : \mathcal{D}(M) \to \mathcal{O}(f)$ defined by $p(h) = f \circ h^{-1}$ is a Serre fibration with fiber $\mathcal{S}(f)$ in topologies $\mathcal{C}^\infty$.*

Recall that if $f : (\mathbb{C}, 0) \to (\mathbb{R}, 0)$ is a smooth germ for which $0 \in \mathbb{C}$ is an *isolated* critical point, then there exists a *homeomorphism* $h : \mathbb{C} \to \mathbb{C}$ such that $h(0) = 0$ and

$$f \circ h(z) = \begin{cases} \pm|z|^2, & \text{if } z \text{ is a } \textit{local extremum}, \text{ [3]}, \\ \mathrm{Re}(z^n), (n \geq 1) & \text{otherwise, so } z \text{ is a } \textit{saddle}, \text{ [15]}, \end{cases}$$

Examples of the foliation by level sets of f near 0 are presented in Figure 1.1.

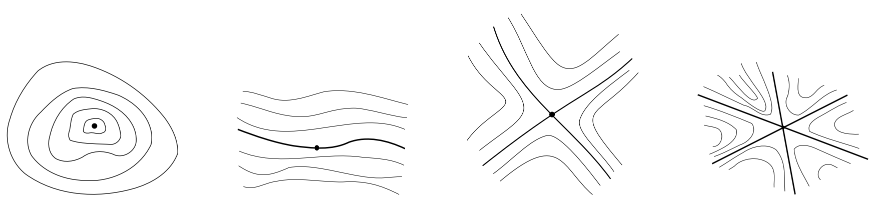

FIGURE 1.1. Isolated critical points

From this point of view S-points are saddles, while P- and N-points a local extremes. Moreover, P-points admit non-trivial f-preserving circle actions (as non-degenerate local extremes do), while N-points admit only $\mathbb{Z}_n$-action preserving f. We will not give precise definitions but recall a large class of examples of such points.

Example 1.1. [10]. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial without multiple factors with* $\deg f \geq 2$*, so*

$$f = L_1 \cdots L_a \cdot Q_1 \cdots Q_b, \qquad a + 2b \geq 2,$$

where every L_i is a linear function and every Q_j is an irreducible over $\mathbb{R}$ (i.e. definite) quadratic form such that $L_i/L_{i'} \neq \mathrm{const}$ for $i \neq i'$ and $Q_j/Q_{j'} \neq \mathrm{const}$ for $j \neq j'$.

If $a \geq 1$, so f has linear factors and thus 0 is a saddle, then the origin $0 \in \mathbb{R}^2$ is an S-point for f.

If $a = 0$ and $b = 1$, so $f = Q_1$, then the origin $0 \in \mathbb{R}^2$ is a P-point for f.

Otherwise, $a = 0$ and $b \geq 2$, so $f = Q_1 \cdots Q_b$. Then the origin $0 \in \mathbb{R}^2$ is an N-point for f.

Lemma 1.2. [10]. *Let $f : M \to P$ be a C^∞ map satisfying* (Bd)*, and such that every of its critical points belongs to the class described in Example 1.1, in particular, f also satisfies* (SPN)*. Then f also satisfies* (Fibr).

It follows from Morse lemma and Example 1.1 that non-degenerate saddles are S-points while non-degenerate local extremes are P-points.

Now the main result of [12] can be formulated as follows.

Theorem 1.3. [10, 12]. *Suppose $f : M \to P$ satisfies* (Bd) *and* (SPN)*. If f has at least one S- or N-point, or if M is non-orientable, then $\mathcal{S}_{\mathrm{id}}(f)$ is contractible.*

Moreover, if in addition f satisfies (Fibr)*, then $\pi_i \mathcal{O}_f(f) = \pi_i M$ for $i \geq 3$, $\pi_2 \mathcal{O}_f(f) = 0$, and for $\pi_1 \mathcal{O}(f)$ we have the following short exact sequence*

$$1 \to \pi_1 \mathcal{D}(M) \oplus \mathbb{Z}^l \to \pi_1 \mathcal{O}_f(f) \to G \to 1,$$

for a certain finite group G and $l \geq 0$ both depending on f.

Thus, the information about the fundamental group $\pi_1 \mathcal{O}_f(f)$ is not complete. The aim of this note is to show that the calculation of $\pi_1 \mathcal{O}_f(f)$ can be reduced to the case when M is either a 2-disk, or a cylinder, or a Möbius band, see Theorems 1.7 and 1.8 below. The obtained results hold for a more general class of maps $M \to P$ than the one considered in [12].

1.4. **Admissible critical points.** We will now introduce a certain type of critical points for f. Let F be a vector field on M, $V \subset M$ be an open subset, and $h : V \to M$ be an embedding. Say that h *preserves orbits of F* if for every orbit o of F we have that $h(V \cap o) \subset o$.

Definition 1.5. *Let $f : M \to P$ be a C^∞ map and $z \in \mathrm{Int} M$ be an isolated critical point of f which is not a local extreme (so z is a saddle). Say that z is **admissible** if there exists a neighbourhood U of z containing no other critical points of f and a vector field F on U having the following properties:*

(1) *f is constant along orbits of F and z is a unique singular point of F.*
(2) *Let $(\mathbf{F}_t)$ be the local flow of F on U. Then for every germ of diffeomorphisms $h : (M, z) \to (M, z)$ preserving orbits of F there exists a C^∞ germ $\sigma : (M, z) \to \mathbb{R}$ such that $h(x) = \mathbf{F}(x, \sigma(x))$ near z.*

This definition almost coincides with the definition of an S-point, c.f. [12]. The difference is that for S-points it is also required that the correspondence $h \mapsto \sigma$ is continuous with respect to topologies C^∞. In particular every S-point is admissible.

Now put the following two axioms for f both implied by (SPN):

(Isol) *All critical points of f are isolated.*

(SA) *Every saddle of f is admissible.*

1.6. **Main result.** Let $\mathcal{D}_{\mathrm{id}}(M)$ be the identity path component of the group $\mathcal{D}(M)$ and

$$\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M)$$

be the stabilizer of f with respect to the right action of $\mathcal{D}_{\mathrm{id}}(M)$. Thus $\mathcal{S}'(f)$ consists of diffeomorphisms h isotopic to id_M and preserving F, i.e. $f \circ h = f$.

For a closed subset $X \subset M$ denote by $\mathcal{S}'(f, X)$ the subgroup of $\mathcal{S}'(f)$ consisting of diffeomorphisms fixed on some neighbourhood of X.

The aim of this note is to prove the following theorem:

Theorem 1.7. *Suppose $\chi(M) < 0$. Let $f : M \to P$ be a C^∞ map satisfying the axioms (Bd), (Isol), and (SA). Then there exists a compact subsurface $X \subset M$ with the following properties:*

(1) f is locally constant on ∂X and every connected component B of $\overline{M \setminus X}$ is either a 2-disk or a 2-cylinder or a Möbius band. Moreover, $\partial B \subset X$ and B contains critical points of f.

(2) Let $h \in \mathcal{S}'(f, X)$ and B be a connected component of $\overline{M \setminus X}$, thus h is fixed on some neighbourhood of ∂B. Then the restriction $h|_B$ is isotopic in B to id_B with respect to some neighbourhood of ∂B.

(3) The inclusion $i : \mathcal{S}'(f, X) \subset \mathcal{S}'(f)$ induces a group isomorphism $i_0 : \pi_0\mathcal{S}'(f, X) \approx \pi_0\mathcal{S}'(f)$.

The proof of this theorem will be given in §7. We will now show how to simplify calculations of $\pi_1\mathcal{O}(f)$ using Theorem 1.7.

Let X be the surface of Theorem 1.7 and let $B_1, \ldots, B_l$ be all the connected components of $\overline{M \setminus X}$. For every $i = 1, \ldots, l$ denote by

$\mathcal{D}_{\mathrm{id}}(B_i, \partial B_i)$ the group of diffeomorphisms of B_i fixed on some neighbourhood of ∂B_i and isotopic to id_{B_i} relatively to some neighbourhood of B_i. Let also $\mathcal{S}'(f|_{B_i}, \partial B_i)$ be the stabilizer of the restriction $f|_{B_i} : B_i \to P$ with respect to the right action of $\mathcal{D}_{\mathrm{id}}(B_i, \partial B_i)$. Then we have an evident isomorphism of groups:

$$\psi : \mathcal{S}'(f, X) \ \approx \ \mathop{\times}_{i=1}^{l} \mathcal{S}'(f|_{B_i}, \partial B_i), \qquad \psi(h) = (h|_{B_1}, \dots, h|_{B_l}), \tag{1.1}$$

It is easy to show that ψ is in fact a homeomorphism with respect to the corresponding C^∞ topologies.

Theorem 1.8. *Under assumptions of Theorem 1.7 suppose that f also satisfies* (Fibr). *Then we have an isomorphism:*

$$\pi_1 \mathcal{O}_f(f) \ \approx \ \mathop{\times}_{i=1}^{l} \pi_0 \mathcal{S}'(f|_{B_i}, \partial B_i).$$

Proof. It is easy to show that if f satisfies (Fibr), then $\mathcal{O}_f(f)$ is the orbit of f with respect to the action of $\mathcal{D}_{\mathrm{id}}(M)$ and the projection $p : \mathcal{D}_{\mathrm{id}}(M) \to \mathcal{O}_f(f)$ is a Serre fibration as well, see [11]. Hence we get the following part of exact sequence of homotopy groups

$$\cdots \to \pi_1 \mathcal{D}_{\mathrm{id}}(M) \to \pi_1 \mathcal{O}_f(f) \to \pi_0 \mathcal{S}'(f) \to \pi_0 \mathcal{D}_{\mathrm{id}}(M) \to \cdots$$

Since $\chi(M) < 0$, we have $\pi_1 \mathcal{D}_{\mathrm{id}}(M) = 0$, [5, 4, 7]. Moreover, $\mathcal{D}_{\mathrm{id}}(M)$ is path-connected, whence together with Theorem 1.7 we obtain an isomorphism:

$$\pi_1 \mathcal{O}_f(f) \ \approx \ \pi_0 \mathcal{S}'(f) \ \overset{i_0}{\approx} \ \pi_0 \mathcal{S}'(f, X) \ \overset{(1.1)}{\approx} \ \mathop{\times}_{i=1}^{l} \pi_0 \mathcal{S}'(f|_{B_i}, \partial B_i).$$

Theorem is proved. □

Thus a general problem of calculation of $\pi_1 \mathcal{O}_f(f)$ for maps satisfying the above axioms completely reduces to the case when $\chi(M) \geq 0$. A presentation for $\pi_1 \mathcal{O}_f(f)$ will be given in another paper.

1.9. **Structure of the paper.** In next four sections we study incompressible subsurfaces $N \subset M$. §2 contains their definition and some elementary properties. In §3 we show how such subsurfaces appear in studying maps $M \to P$ with isolated singularities. In §4 and §5 we extend results of W. Jaco and P. Shalen [8] about deformations of incompressible subsurfaces and periodic automorphisms of surfaces. §6 contains two technical statements about deformations of diffeomorphisms preserving a map $M \to P$. Finally in §7 we prove Theorem 1.7.

2. Incompressible subsurfaces

The following Lemma 2.1 is well-known, see e.g. [14, Pr. 2.1]. It was also implicitly formulated in [8, page 359].

Lemma 2.1. 1) *Let M be a connected surface, and $N \subset \mathrm{Int}M$ be a proper compact (possibly not connected) subsurface neither of whose connected components is a 2-disk. Then the following conditions are equivalent:*

(a) *for every connected component N_i of N the inclusion homomorphism $\pi_1 N_i \to \pi_1 M$ is injective;*
(b) *none of the connected components of $\overline{M \setminus N}$ is a 2-disk.*

If these conditions hold, then N will be called ***incompressible****, see* [8, Def. 3.2].

Corollary 2.2. *If $N \subset M$ is incompressible, then $\chi(M) \le \chi(N)$.*

Corollary 2.3. *Let $R \subset \mathrm{Int}M$ be a proper compact connected subsurface. Then the following conditions are equivalent:*

(R1) *the homomorphism $\xi : \pi_1 R \to \pi_1 M$ is trivial;*
(R2) *R is contained in some 2-disk $D \subset M$.*

Proof. The implication (R2)⇒(R1) is evident.

(R1)⇒(R2). Suppose R is not contained in any 2-disk. We will show that ξ is non-trivial. Let N be the union of R with all of the connected components of $\overline{M \setminus N}$ which are 2-disks. Then by our assumption N is not a 2-disk and by Lemma 2.1 N is incompressible. Notice that ξ is a product of homomorphisms induced by the inclusions $R \subset N \subset M$:

$$\xi = \beta \circ \alpha : \pi_1 R \stackrel{\alpha}{\to} \pi_1 N \stackrel{\beta}{\to} \pi_1 M.$$

Also notice that α is surjective and by Lemma 2.1 β is a non-trivial monomorphism. Hence ξ is also non-trivial. □

Corollary 2.4. *Let $R \subset \mathrm{Int}M$ be a proper (possibly non connected) subsurface such that neither of its connected components is contained in some 2-disk. Then every connected component B of $\overline{M \setminus R}$ which is not a 2-disk is incompressible.*

Proof. Let C be a connected component of $\overline{M \setminus B}$. Due to Lemma 2.1 it suffices to show that C is not a 2-disk. Notice that $C \cap R \neq \varnothing$, whence it contains some connected component R_i of R. By Corollary 2.3 the product of homomorphisms $\pi_1 R_i \to \pi_1 C \to \pi_1 M$ is non-trivial, and therefore $\pi_1 C \to \pi_1 M$ is also non-trivial. This implies that C is not a 2-disk. □

3. Incompressible subsurfaces associated to a map $M \to P$

3.1. **Singular foliation Δ_f of f.** Let $f : M \to P$ be a map satisfying axioms (Bd) and (Isol). Then f induces on M a one-dimensional foliation Δ_f with singularities defined as follows: *a subset $\omega \subset M$ is a leaf of Δ_f if and only if ω is either a critical point of f or a connected component of the set $f^{-1}(c) \setminus \Sigma_f$ for some $c \in P$.* Thus the leaves of Δ_f are 1-dimensional submanifolds of M and critical points of f. Local structure of Δ_f near critical points of f is illustrated in Figure 1.1.

Denote by Δ_f^{reg} the union of all leaves of Δ_f homeomorphic to the circle and by Δ_f^{cr} the union of all other leaves. The leaves in Δ_f^{reg} (resp. Δ_f^{cr}) will be called *regular* (resp. *critical*). Similarly, connected components of Δ_f^{reg} (resp. Δ_f^{cr}) will be called *regular* (resp. *critical*) components of Δ_f. It follows from (Bd) that $\partial M \subset \Delta_f^{\mathrm{reg}}$. It is also evident, that every critical leaf of Δ_f^{cr} either is homeomorphic to an open interval or is a critical point of f.

3.2. **Atoms and canonical neighbourhoods of critical components of Δ_f.** For every critical component K of Δ_f define its regular neighbourhood R_K as follows. Let $c_1, \ldots, c_l$ be all the critical values of f and the values of f on ∂M. Since M is compact, it follows from axioms (Bd) and (Isol) that l is finite. For each $i = 1, \ldots, l$ let $W_i \subset P$ be a closed connected neighbourhood (i.e. just an arc) of c_i containing no other c_j. We will assume that $W_i \cap W_j = \varnothing$ for $i \neq j$.

Now let K be a critical component of Δ_f. Then $f(K) = c_i$ for some i. Let R_K be the connected component of $f^{-1}(W_i)$ containing K. Evidently, R_K is a union of leaves of Δ_f. Following [2] we will call R_K an *atom* of K, see Figure 3.1.

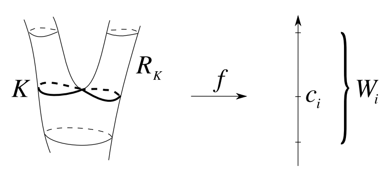

Figure 3.1.

Evidently, R_K is a regular neighbourhood of K with respect to some triangulation of M. Similarly to [8] define the *canonical neighbourhood* N_K of K to be the union of R_K with all the connected components of $\overline{M \setminus R_K}$ being 2-disks. If N_K is not a 2-disk, then by Lemma 2.1 N_K is incompressible in M.

Notice that

$$\partial R_K \;=\; f^{-1}(\partial W_i) \;\cap\; R_K. \tag{3.1}$$

Let K' be another critical component of Δ_f such that $f(K') = f(K)$. Since $R_{K'}$ is also constructed via W_i, we obtain from (3.1) that f *takes on $\partial R_{K'}$ the same values as on ∂R_K.* This technical assumption is not essential, however it will be useful for the proof of Theorem 1.7.

Lemma 3.3. *Let K and K' be two distinct critical components of Δ_f.*

(i) *Then $R_K \cap R_{K'} = \varnothing$, while N_K and $N_{K'}$ are either disjoint or one of them, say N_K, is contained in $N_{K'}$. In the last case N_K is a 2-disk.*

(ii) *Suppose $f(K) = f(K')$ and there exists $h \in \mathcal{S}(f)$ such that $h(K) = K'$. Then $h(R_K) = R_{K'}$ and $h(N_K) = N_{K'}$.*

Proof. (i) follows from the assumption that $W_i \cap W_j = \varnothing$ for $i \neq j$, and (ii) follows from (3.1). We leave the details for the reader. □

Lemma 3.4. *Let K be a critical component of Δ_f such that N_K is a 2-disk. Then either*

(i) *M is a 2-disk itself, or*

(ii) *N_K is contained in a unique canonical neighbourhood $N_{K'}$ of another critical component K' of Δ_f such that $N_{K'}$ is not a 2-disk.*

Proof. Let $\mathbf{R}$ be the union of atoms of all critical components of Δ_f. Then every connected component B of $\overline{M \setminus \mathbf{R}}$ is diffeomorphic to the cylinder $S^1 \times [0,1]$ and the restriction $f|_B$ has no critical points.

Notice that $\overline{M \setminus N_K}$ is connected since N_K is a 2-disk. Also, there exists a unique connected component B (being a cylinder $S^1 \times [0,1]$) of $\overline{M \setminus \mathbf{R}}$ such that $\partial N_K \subset B$. Then $N_K \cup B$ is also a 2-disk.

Let n be the total number of critical components of Δ_f in $\overline{M \setminus N_K}$.

If $n = 0$, then $N_K \cup B = M$. Whence M is a 2-disk.

Suppose that $n \geq 1$. Let γ be another connected component of ∂B distinct from ∂N_K. Then there exists an atom $R_{K'}$ of some critical component K' of Δ_f such that $\gamma \subset \partial R_{K'}$. Since $N_K \cup B$ is a 2-disk, we see that it is contained in $N_{K'}$. If $N_{K'}$ is not a 2-disk, then the lemma is proved. Otherwise, the number of critical components in $\overline{M \setminus N_{K'}}$ is less than in $\overline{M \setminus N_K}$ and the lemma holds by the induction on n. □

Example 3.5. Let $\mathbb{T}^2$ be a 2-torus embedded in $\mathbb{R}^3$ as shown in Figure 3.2 and $f : \mathbb{T}^2 \to \mathbb{R}$ be the projection onto the vertical line. Figure 3.2a) shows the critical components of level-sets of f, and Figure 3.2b) presents blackened canonical neighbourhoods of three critical components of Δ_f containing canonical neighbourhoods of all other critical components of Δ_f.

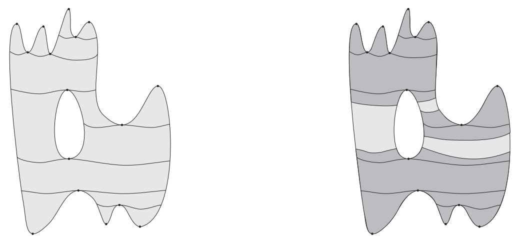
a) b)

FIGURE 3.2.

3.6. **Canonical neighbourhoods of negative Euler characteristic.** Suppose M is not a 2-disk. Let $K_1, \dots, K_r$ be all the critical components of Δ_f whose canonical neighbourhoods are not 2-disks. By Lemma 3.4 this collection is non-empty and by Lemma 3.3 $N_{K_i} \cap N_{K_j} = \varnothing$ for $i \neq j$. Moreover, again by Lemma 3.4, any other critical component of Δ_f is contained in some N_{K_i}. It follows that $\overline{M \setminus \cup_{i=1}^{r} N_{K_i}}$ contains no critical points of f, whence it is a disjoint union of cylinders $S^1 \times I$. Therefore

$$\chi(M) = \sum_{i=1}^{r} \chi(N_{K_i}). \tag{3.2}$$

The following two statements will be used for the construction of a surface X of Theorem 1.7, see §7.

Lemma 3.7. *The following conditions are equivalent:*

(1) $\chi(M) < 0$*;*
(2) $\chi(N_{K_i}) < 0$ *for some* $i = 1, \dots, r$*.*

Proof. (1)⇒(2). As $\chi(M) < 0$, we get from (3.2) that $\chi(N_{K_i}) < 0$ for some i.

The implication (2)⇒(1) follows from Corollary 2.2. □

Corollary 3.8. *Let $K_1, \dots, K_k$ be all the critical components of Δ_f whose canonical neighbourhoods have negative Euler characteristic and $R_{K_1}, \dots, R_{K_k}$ be their atoms. Put $\mathcal{R}_{<0} := \cup_{i=1}^{k} R_{K_i}$. If $\mathcal{R}_{<0} \neq \varnothing$, then every connected component B of $\overline{M \setminus \mathcal{R}_{<0}}$ is either a 2-disk, or a cylinder, or a Möbius band.*

Proof. Since the homomorphism $\pi_1 R_{K_i} \to \pi_1 M$ is non-trivial for each i, it follows from Corollary 2.4 that B is incompressible. Suppose $\chi(B) < 0$. Notice that f takes constant values of ∂B. Then by Lemma 3.7 there exists a critical component $K \subset B$ of Δ_f such that the canonical neighbourhood N of K with respect to $f|_B$ has negative Euler

characteristic. It follows that the homomorphisms $\pi_1 N \to \pi_1 B \to \pi_1 M$ induced by the inclusions $N \subset B \subset M$ are monomorphisms, so N is incompressible in M. This implies that N is a canonical neighbourhood of K with respect to f. But since $\chi(N) < 0$, we should have that $N \subset \mathcal{R}_{<0}$, which contradicts to the assumption. $\square$

4. Deformations of incompressible subsurfaces

The aim of this section is to extend some results of [8] concerning incompressible subsurfaces, see Proposition 4.5.

4.1. **$\pm$-twist.** Let $\gamma \subset \mathrm{Int}M$ be a two-sided simple closed curve, U be its regular neighbourhood diffeomorphic to $S^1 \times [-1,1]$ so that γ correspond to $S^1 \times 0$. Take a function $\mu : [-1,1] \to [0,1]$ such that $\mu = 0$ near $\{\pm 1\}$ and $\mu = 1$ on some neighbourhood of 0. Define the following homeomorphism $g_\gamma : M \to M$ by

$$g_\gamma(x) = \begin{cases} (z\, e^{2\pi i \mu(t)}, t), & x = (z,t) \in S^1 \times [-1,1] \cong U \\ x, & x \in M \setminus U, \end{cases} \tag{4.1}$$

see Figure 4.1. Then g_γ is fixed on some neighbourhood of $\overline{M \setminus U}$ and isotopic to id_M via an isotopy supported in $\mathrm{Int}U$. Evidently, g_γ is a product of Dehn twists in opposite directions along the curves parallel to γ. Therefore we will call g_γ a *$\pm$-twist near γ*.

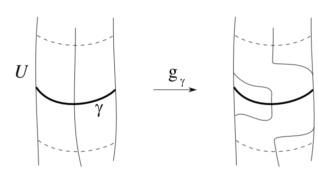

Figure 4.1. $\pm$-twist

The following lemma is a particular case of [6, Lm. 6.1].

Lemma 4.2. [6, Lm. 6.1]. *Suppose $\chi(M) < 0$. Let $\gamma \subset \mathrm{Int}M$ be a simple closed curve which does not bound a 2-disk nor a Möbius band, $h : M \to M$ be a homeomorphism homotopic to id_M and such that $h(\gamma) = \gamma$. Let also $H : M \times I \to M$ be any homotopy of id_M to h. Then there exists another homotopy $G_t : M \times I \to M$ of id_M to h such that $G_t(\gamma) = \gamma$ and $G_t = H_t$ on $\overline{M \setminus U}$ for all $t \in I$.*

Moreover, there exists $m \in \mathbb{Z}$ and a homotopy $G' : M \times I \to M$ of id_M to $g_\gamma^m \circ h$ such that $G'_t = G$ outside U and G'_t is fixed on γ for all $t \in I$.

The following statement is also well-known.

Lemma 4.3. *Let M be a surface with $\chi(M) < 0$. Suppose $\partial M \neq \varnothing$ and let $\gamma_1, \ldots, \gamma_l$ be all the connected components of ∂M. For each $i = 1, \ldots, l$ let τ_i be a Dehn twist along the curve parallel to γ_i and fixed on ∂M. Let $m_1, \ldots, m_l \in \mathbb{Z}$ be integer numbers not of all are equal to zero. Then the homeomorphism $\tau_1^{m_1} \circ \cdots \circ \tau_l^{m_l}$ is **not homotopic to** id_M via a homotopy fixed on ∂M.*

4.4. **Deformations of incompressible subsurfaces.** Let M be a surface distinct from the 2-sphere S^2 and the projective plane $\mathbb{R}\mathrm{P}^2$, $N \subset M$ be an incompressible subsurface, and $N_1, \ldots, N_k$ be all of its connected components. Let also $h : M \to M$ be a homeomorphism homotopic to id_M and $H : M \times I \to M$ be any homotopy of id_M to h.

The following Proposition 4.5 follows the line of [8, Lm. 4.2]. In fact the first part of statement (B) is a particular case of that lemma.

Proposition 4.5. c.f. [8, Lm. 4.2] (A) *If N_j is not a cylinder for some j, then $h(N_j) \cap N_j \neq \varnothing$.*

(B) *Suppose $\chi(N_j) < 0$ and $h(N_j) \subset N_j$ for some j. Then there exists a homotopy $G : N_j \times I \to N_j$ of the identity map id_{N_j} to the restriction $h|_{N_j}$ such that $G_t(x) = H_t(x)$ whenever $H(x \times I) \subset N_j$.*

Moreover, suppose $H(\gamma \times I) \subset \gamma$ for each connected component γ of ∂N_j. Extend G to a map $G : M \times I \to M$ by $G_t = H_t$ on $M \setminus N_j$. Then G is a homotopy of id_M to h.

(C) *Suppose $\chi(N_j) < 0$ and $h(N_j) = N_j$ for all $j = 1, \ldots, k$. Then there exists a homotopy $G : M \times I \to M$ of id_M to h such that $G(N_j \times I) \subset N_j$ for all $j = 1, \ldots, k$ and $G(B \times I) \subset B$ for every connected component B of $\overline{M \setminus N}$.*

(D) *Suppose $\chi(N_j) < 0$ and h is fixed on N for all $j = 1, \ldots, k$. Then there exists a homotopy of id_M to h fixed on N.*

Proof. First we make the following remark which repeats the key arguments of [8, Lm. 4.2]. For $j = 1, \ldots, k$ let $p_j : \widetilde{M}_j \to M$ be the covering map corresponding to the subgroup $\pi_1 N_j$ of $\pi_1 M$. Then the embedding $i : N_j \subset M$ lifts to the embedding $i^* : N_j \to \widetilde{M}_j$ which induces an isomorphism between $\pi_1 N_j$ and $\pi_1 \widetilde{M}_j$. Denote $\widetilde{N}_j = i^*(N_j)$. Then we have the following commutative diagram:

$$\begin{tikzcd} \widetilde{N}_j \arrow[r, hook] & \widetilde{M}_j \arrow[d, "p_j"] \\ N_j \arrow[u, "\cong"] \arrow[ur, "i^*"] \arrow[r, "i"] & M \end{tikzcd}$$

Since $\widetilde{M}_j$ and N_j are aspherical, it follows from Whitehead's theorem that $\widetilde{N}_j$ is a strong deformation retract of $\widetilde{M}_j$. Then every connected component of $\mathrm{Int}(\widetilde{M}_j \setminus \widetilde{N}_j)$ is an open cylinder. Let $H : N_j \times I \to M$ be any homotopy between the identity embedding $H_0 = i : N_j \subset M$ and $H_1 = h|_{N_j}$. Then there exists a lifting $\widetilde{H} : N_j \times I \to \widetilde{M}_j$ such that $\widetilde{H}_0 = i^*$ and $p_j \circ \widetilde{H} = H$. Denote $\widetilde{N}'_j = \widetilde{H}_1(N_j)$. Since both $\widetilde{N}_j$ and $\widetilde{N}'_j$ are deformational retracts of $\widetilde{M}_j$, they are incompressible in $\widetilde{M}_j$.

(A) Suppose $h(N_j) \cap N_j = \varnothing$. Then $\mathrm{Int}(\widetilde{N}'_j)$ is included into some connected component C of $\mathrm{Int}(\widetilde{M}_j \setminus \widetilde{N}_j)$ being a cylinder. Since $\widetilde{N}'_j$ is incompressible in M, it is also incompressible in C, whence $\widetilde{N}'_j$ and therefore N_j are cylinders. Thus if N_j is not a cylinder, then we obtain that $h(N_j) \cap N_j \neq \varnothing$.

(B) Let $r_j : \widetilde{M}_j \to \widetilde{N}_j$ be any retraction. Then the map

$$G = p_j \circ r_j \circ \widetilde{H} \ : \ N_j \times I \ \to \ N_j$$

is a homotopy of id_{N_j} to $h|_{N_j}$ in N_j. It is easy to see that $G_t(x) = H_t(x)$ whenever $H(x \times I) \subset N_j$.

Suppose that $H(\gamma \times I) \subset \gamma \subset N_j$ for each connected component γ of ∂N_j. Then by the construction $G_t = H_t$ on ∂N_j. Notice that ∂N_j separates M. Extend G to all of $M \times I$ by $G = H$ of $(M \setminus N_j) \times I$. Then G is continuous, $G_0 = \mathrm{id}_M$ and $G_1 = h$.

(C) Suppose $\chi(N_j) < 0$ and $h(N_j) = N_j$ for all $j = 1, \dots, k$. Let $\gamma_1, \dots, \gamma_l$ be all the connected components of ∂N. Since N is incompressible, we have by Corollary 2.2 that $\chi(M) \leq \chi(N_j) < 0$ as well. Moreover, by (B) for each j the restriction $h|_{N_j}$ is a homeomorphism of N_j homotopic in N_j to id_{N_j}. This, in particular, implies that $h(\gamma_i) = \gamma_i$ for $i = 1, \dots, l$.

Then by Lemma 4.2 we can suppose that $H(\gamma_i \times I) \subset \gamma_i$ for all $i = 1, \dots, l$ as well. Moreover, due to (B) it can be additionally assumed that $H(N_j \times I) \subset N_j$.

Let B be a connected component of $\overline{M \setminus N}$. Since N is incompressible, B is not a 2-disk. Then by Corollary 2.4 B is incompressible. Therefore we can apply statement (B) to B and change the homotopy G on $B \times I$ so that $G(B \times I) \subset B$.

(D) Suppose h is fixed on N. For each i let U_i be a regular neighbourhood of γ_i, and g_i be a $\pm$-twist near γ_i supported in U_i. We can assume that $U_i \cap U_j = \varnothing$ for $i \neq j$. Then by Lemma 4.2 there exist integer numbers $m_1, \dots, m_l \in \mathbb{Z}$ and a homotopy $G : M \times I \to M$ of

id_M to a homeomorphism $h' := g_1^{m_1} \circ \cdots \circ g_l^{m_l} \circ h$ such that G_t is fixed on L for each $t \in I$. By (C) we can also assume that $G(N_j \times I) \subset N_j$ and $G(B \times I) \subset B$ for every connected component of $\overline{M \setminus N}$ and each $j = 1, \dots, k$.

In particular, we see that the restriction $h'|_N$ is homotopic to id_N relatively ∂N. But this restriction is evidently a product of Dehn twists along boundary components of N. Since $\chi(N_j) < 0$ for all j, we get from Lemma 4.3 that $m_i = 0$ for all $i = 1, \dots, l$. Hence $h' = h$. Thus G is in fact a homotopy between id_M and h relatively ∂N. Since ∂N separates M, and id_M and h are fixed on N, we can change G on $N \times I$ by $G_t(x) = x$. This gives a homotopy between id_M and h relatively to N. □

5. Automorphisms of cellular subdivisions

Let N be a compact surface and $\Xi = \{e_\lambda\}_{\lambda \in \Lambda}$ be some partition of N into a disjoint family of connected orientable submanifolds. Say that a homeomorphism $h : N \to N$ is a Ξ-*homeomorphism* provided it yields a permutation of elements of Ξ, that is for each $e \in \Xi$ its image $h(e)$ also belongs to Ξ. An element $e \in \Xi$ will be called h-*invariant* if $h(e) = e$. Say that e is h^+-*invariant* (h^--*invariant*) if the restriction $h|_e : e \to e$ is a preserving (reversing) orientation map. We will also say that h is Ξ-*trivial* if each $e \in \Xi$ is h^+-invariant.

Remark 5.1. Notice that we can say that a map $h : e \to e$ preserves or reverses orientation only if $\dim e \geq 1$. To each 0-dimensional element $e \in \Xi$ (being of course a point) we formally assign a "positive orientation" and assume that *by definition every cellular homeomorphism preserves orientation of each invariant* 0-*element of* Ξ.

Example 5.2. Let M be a connected surface and $K \subset \mathrm{Int}M$ be an embedded finite connected graph. Assume that K is a subcomplex of M with respect to some triangulation of M. By R_K we will denote a regular neighbourhood of K. Following [8] define a *canonical* neighbourhood N_K of K to be the union of a regular neighbourhood R_K of K with those connected components of $M \setminus R_K$ which are 2-disks. Notice that $N_K \setminus K$ is a disjoint union of open 2-disks and half-open cylinders $S^1 \times (0, 1]$ with $S^1 \times \{1\}$ corresponding boundary components of ∂N_K. Thus we obtain a natural partition of N_K by vertexes and edges of K and connected components of $N_K \setminus K$. We shall denote this partition by Ξ_K.

Now let Ξ be a cellular subdivision of N. Denote by N_i ($i = 0, 1, 2$) the i-th skeleton of N. In particular, N_1 is a finite connected subgraph

in N such that $N \setminus N_1$ is a disjoint union of 2-disks. Let c_i $(i = 0, 1, 2)$ be the total number of i-cells of Δ. Then of course $\chi(N) = c_0 - c_1 + c_2$.

Let $C = \{C_i, \partial_i\}$ be the $\mathbb{R}$-chain complex of N corresponding to a given cellular subdivision. Thus C_i is a real vector space of dimension c_i with the *oriented* i-cells of Ξ as a basis. Then every Ξ-homeomorhism h induces a chain automorphism $\{h_i : C_i \to C_i,\ i = 0, 1, 2\}$ of C.

Recall that for each continuous mapping $h : N \to N$ we can define its *Lefschetz number* $L(h)$ by the formula:

$$L(h) = \operatorname{tr}(\bar{h}_0) - \operatorname{tr}(\bar{h}_1) + \operatorname{tr}(\bar{h}_2),$$

where $\bar{h}_i : H_i(N, \mathbb{R}) \to H_i(N, \mathbb{R})$ is the induced homomorphism of the corresponding homology groups and tr is the trace of this homomorphism. If h is cellular, then $L(h)$ can also be calculated via the chain homomorphisms h_i by:

$$L(h) = \operatorname{tr}(h_0) - \operatorname{tr}(h_1) + \operatorname{tr}(h_2).$$

The following theorem is relevant to [8, Lm. 4.4] being a statement about periodic homeomorphisms.

Theorem 5.3. c.f. [8, Lm. 4.4]. *Let M be a compact surface, $K \subset M$ a connected subgraph, N_K be a canonical neighbourhood of K. Let also $h : M \to M$ a homeomorphism such that h is homotopic to id_M, $h(K) = K$, and h preserves the set of vertexes of K of degree 2, and $h(N_K) = N_K$. In particular, $h|_{N_K}$ is a Ξ_K-homeomorphism.*

1. *If $\chi(N_K) < 0$, then h is Ξ_K-trivial.*
2. *Suppose that $N_K = M$, M is orientable, and $\chi(M) \geq 0$. Then every **annulus** $a \in \Xi_K$ is h^+-invariant, and the total number of h-invariant **cells** of Ξ_K is equal to $\chi(M)$.*

The proof of Theorem 5.3 will be given in §5.7. It is based on Proposition 4.5 and on the following statement.

Proposition 5.4. *Let N be a **closed, orientable** surface endowed with some cellular subdivision Ξ and $h : N \to N$ be a Ξ-homeomorphism **preserving orientation** of N and being not Ξ-trivial, i.e. $h(e) \neq e$ for some cell $e \in \Xi$. Then the number of h-invariant cells of Ξ is precisely equal to $L(h)$. In particular, $L(h) \geq 0$.*

Proof. Let k_i $(i = 0, 1, 2)$ be the number of h-invariant i-cells of Ξ and $k := k_0 + k_1 + k_2$. We will show that

$$k_i = (-1)^i \operatorname{tr}(h_i), \tag{5.1}$$

which will imply

$$k = \sum_{i=0}^{2} k_i = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(h_i) = L(h).$$

To prove (5.1) we have to show that there are no h^--invariant 0- and 2-cells and no h^+-invariant 1-cells. For 0-cells this holds by Remark 5.1 and for 2-cells from the assumptions that N is orientable and h preserves orientation.

Let e be an h-invariant 1-cell and f_0 and f_1 be two 2-cells that are incident to e. It is possible of course that $f_0 = f_1$. Since h preserves orientation, it follows that

(a) either $h_2(f_j) = +f_j$ for $j = 0, 1$, and $h_1(e) = +e$,

(b) or $h_2(f_j) = +f_{1-j}$ for $j = 0, 1$, and $h_1(e) = -e$.

The following Claim 5.5 implies that in the case (a) h is Ξ-trivial. Since h is not Ξ-trivial, we will get from (b) that all h-invariant 1-cells are h^--invariant.

Claim 5.5. *Suppose that there exists a* 1*-cell* $e \in \Xi$ *such that*

(i) $h_1(e) = +e \in C_1$ *and*

(ii) h *preserves each* 2*-cell which is adjacent to* e*.*

Then h *is* Ξ*-trivial.*

Proof. Notice that for each vertex $v \in N_0$ the inclusion $N_1 \subset N$ induces a cyclic ordering of edges that are incident to v.

Let v be a vertex of e. Then it follows from (i) and (ii) that all of the 1- and 2-cells incident to v are h^+-invariant. Moreover, for each 1-cell that is incident to v the conditions (i) and (ii) also hold true. Since N is connected, it follows that h is Ξ-trivial. $\square$

Proposition 5.4 is completed. $\square$

Corollary 5.6. *Let* N *be a* ***closed*** *surface,* Ξ *be a cellular subdivision of* M*, and* $h : N \to N$ *be a* Ξ*-homeomorphism. If* h *is isotopic to* id_N*, then each of the following conditions implies that* h *is* Ξ*-trivial:*

(1) $\chi(N) < 0$*;*

(2) $\chi(N) \geq 0$ *and the total number of* h^+*-invariant* 2*-cells is greater than* $\chi(N)$*.*

Proof. Since h is isotopic to id_N, we have that $L(h) = L(\mathrm{id}_N) = \chi(N)$.

If N is orientable, then h preserves orientation and by Proposition 5.4 h is either Ξ-trivial or has exactly $\chi(N) \geq 0$ invariant cells. Each of the conditions (1) and (2) implies that the number of h-invariant cells is not equal to $\chi(N)$. Hence h is Ξ-trivial.

Suppose that N is non-orientable and let $p : \widetilde{N} \to N$ be its oriented double covering. Then Ξ lifts to some cellular subdivision $\widetilde{\Xi}$ of $\widetilde{N}$ and h lifts to a unique $\widetilde{\Xi}$-cellular homeomorphism $\tilde{h}$ of $\widetilde{N}$ which is isotopic to $\mathrm{id}_{\widetilde{N}}$. Therefore $L(\tilde{h}) = L(\mathrm{id}_{\widetilde{N}}) = \chi(\widetilde{N}) = 2\chi(N)$.

We claim that every of the conditions (1) and (2) implies that $\tilde{h}$ is $\widetilde{\Xi}$-trivial, whence h will be Ξ-trivial.

(1) If $\chi(N) < 0$, then $\chi(\widetilde{N}) < 0$, whence $\tilde{h}$ is $\widetilde{\Xi}$-trivial.

(2) Suppose that $\chi(N) \geq 0$ and the total number b of h^{+}-invariant 2-cells is greater than $\chi(N)$. Let e be an h^{+}-invariant 2-cell of Ξ and $\tilde{e}_1$ and $\tilde{e}_2$ be its liftings in $\widetilde{\Xi}$. Then they are $\tilde{h}^{+}$-invariant. Hence $\tilde{h}$ has at least $2b > 2\chi(N) = \chi(\widetilde{N})$ invariant cells. Then by Proposition 5.4 $\tilde{h}$ is $\widetilde{\Xi}$-trivial. $\square$

5.7. **Proof of Theorem 5.3.** Let $h : M \to M$ be a homeomorphism homotopic to the identity and such that $h|_{N_K}$ is a Ξ_K-homeomorphism. Let $\gamma_1, \dots, \gamma_b$ be all the connected components of ∂N_K, and $a_1, \dots, a_b$ be the annuli of Ξ_K corresponding to them, so that $\gamma_i \subset a_i$. Shrink every γ_i to a point x_i and denote the obtained surface by $\widehat{N}_K$. Then $\widehat{N}_K$ is a closed orientable surface and Ξ_K yields an evident cellular partition $\widetilde{\Xi}$ of $\widehat{N}_K$ such that each annulus a_i corresponds to a certain 2-cell $\widehat{a}_i \in \widetilde{\Xi}$.

Also notice that $\chi(\widehat{N}_K) = \chi(N_K) + b$.

Claim 5.8. *Suppose that either $\chi(N_K) < 0$ or $N_K = M$. Then*

(a) *$h|_{N_K}$ is homotopic to id_{N_K} in N_K.*
(b) *$h(\gamma_i) = \gamma_i$ for $i = 1, \dots, b$ and h preserves orientation of γ_i;*
(c) *h induces some $\widetilde{\Xi}$-homeomorphism $\widehat{h} : \widehat{N}_K \to \widehat{N}_K$ homotopic to $\mathrm{id}_{\widehat{N}_K}$ with respect to $\{x_1, \dots, x_b\}$, in particular, every 2-cell $a_i \in \widetilde{\Xi}$ is $\widehat{h}^{+}$-invariant;*
(d) *$L(\widehat{h}) = L(\mathrm{id}_{\widehat{N}_K}) = \chi(\widehat{N}_K) = \chi(N_K) + b$.*

Proof. (a) For $N_K = M$ this statement is trivial. If $\chi(N_K) < 0$, then by (B) of Proposition 4.5 (or directly by [8, Lm. 4.1]) $h|_{N_K}$ is homotopic to id_{N_K} in N_K. All other statements (b)-(d) follow from (a). $\square$

Now we can complete Theorem 5.3.

(1) Suppose that $\chi(N_K) < 0$. If also $\chi(\widehat{N}_K) < 0$, then by (1) of Corollary 5.6 $\widehat{h}$ is $\widetilde{\Xi}$-trivial, whence h is Ξ_K-trivial as well.

Let $\chi(\widehat{N}_K) \geq 0$. By Claim 5.8 $\widehat{h}$ has at least b $\widehat{h}^{+}$-invariant 2-cells $a_1, \dots, a_b$. Moreover, since $\chi(\widehat{N}_K) - b = \chi(N_K) < 0$, we obtain that

$b > \chi(\widehat{N}_K)$, whence by (2) of Corollary 5.6 $\widehat{h}$ is $\widetilde{\Xi}$-trivial. Therefore h is Ξ_K-trivial.

(2) Suppose that $N_K = M$ and M is orientable. It follows from (c) of Claim 5.8 and Proposition 5.4 that $\widehat{h}$ is either $\widetilde{\Xi}$-trivial or has exactly $\chi(\widehat{N}_K)$ invariant cells. Therefore, h is either Ξ_K-trivial or has exactly $\chi(\widehat{N}_K) - b = \chi(N_K) = \chi(M)$ invariant cells.

6. Deformations of diffeomorphism near critical components of Δ_f

The following two propositions will be crucial for the proof of Theorem 1.7. Suppose $f : M \to P$ satisfies (Bd), (Isol), and (SA).

Proposition 6.1. *Let K be a critical component of Δ_f such that every $z \in K \cap \Sigma_f$ is admissible, R be its atom, and U be any neighbourhood of R. Let also $h \in \mathcal{S}(f)$. Suppose that $h(\omega) = \omega$ for each leaf ω of Δ_f contained in K and that h preserves orientation of ω whenever $\dim \omega = 1$. Then h is isotopic in $\mathcal{S}(f)$ to a diffeomorphism $h' \in \mathcal{S}(f)$ such that $h' = h$ on $M \setminus U$, and h' is the identity on some neighbourhood of R in U.*

Proof. This proposition follows the line of [10, Th. 6.2]. For the convenience of the reader we will recall the key arguments for the case when M is orientable. A non-orientable case can be deduced from the orientable one similarly to [10, Th. 6.2].

As M is orientable, it has a symplectic structure. Let H be the Hamiltonian vector field of f. Then f is constant along orbits of H, the set of singular points of H coincides with the set of critical points of f, and the foliation by orbits of H coincides with Δ_f. In particular, H is tangent to ∂M and therefore generates a flow $\mathbf{H} : M \times \mathbb{R} \to M$.

We will now change H on neighbourhoods of admissible critical points of f similarly to [10, Lm. 5.1]. Let $z \in \Sigma_f$ be such a point and F_z be a vector field on some neighbourhood U_z of z satisfying assumptions of Definition 1.5. Then it follows from (i) of Definition 1.5 that for every $x \in U_z$ the vectors $H(x)$ and F_z are parallel each other. Therefore, using partition unity technique and changing (if necessary) the signs of F_z, we can change H near each $z \in R \cap \Sigma_f$ and assume that $H = F_z$ on U_z.

Claim 6.2. *There exists a neighbourhood U of R and a unique C^∞ function $\sigma : U \to \mathbb{R}$ such that $h(x) = \mathbf{H}(x, \sigma(x))$ for all $x \in U$.*

Proof. Let $z \in K \cap \Sigma_f$. By assumption h preserves leaves of Δ_f (i.e. orbits of $\mathbf{H}$) in K with their orientations. Since $F_z = H$ near z, it follows from (ii) of Definition 1.5 that there exists a neighbourhood V_z of

z and a unique C^∞ function $\sigma_z : V_z \to \mathbb{R}$ such that $h(x) = \mathbf{H}(x, \sigma_z(x))$. Then the functions $\{\sigma_z\}_{z \in K \cap \Sigma_f}$ yield a unique C^∞ function σ on the union $\bigcup\limits_{z \in K \cap \Sigma_f} V_z$. It remains to note that $K \setminus \Sigma_f$ is a disjoint union of open intervals, whence σ uniquely extends to a C^∞ function on R such that $h(x) = \mathbf{H}(x, \sigma(x))$, see [10, Lm. 6.4] for details. □

Then the desired isotopy of h to h' in $\mathcal{S}(f)$ can be constructed similarly to [10, Lm. 4.14]. Take any C^∞ function $\mu : M \to [0, 1]$ such that $\mu = 0$ on some neighbourhood of $\overline{M \setminus U}$, $\mu = 1$ on R, and μ is constant along orbits of F. Then the function $\nu = \mu\sigma$ is C^∞ and well-defined on all of M. Consider the following homotopy

$$g : M \times I \to M, \qquad g_t(x) = \mathbf{F}(x, t\nu(x)). \tag{6.1}$$

Then $g_0 = \mathrm{id}_M$, g_t is fixed on $\overline{M \setminus U}$, and $g_1 = h$ on R. Since μ is constant along orbits of F and h is a diffeomorphism, it follows from [10, Lm. 4.14] that g is an isotopy. Hence $g_t^{-1} \circ h : M \to M$, $(t \in I)$, is an isotopy in $\mathcal{S}(f)$ supported in U and deforming h to a desired diffeomorphism $h' = g_1^{-1} \circ h$. □

Proposition 6.3. *Let $X \subset M$ be a compact subsurface such that ∂X consists of (regular) leaves of Δ_f. Suppose $h \in \mathcal{S}_{\mathrm{id}}(f)$ is fixed on some neighbourhood U of X. Then there exists an isotopy of h to id_M in $\mathcal{S}(f)$ fixed on some neighbourhood of X.*

Proof. Again we will consider only the case when M is orientable. Let $\mathbf{H} : M \times \mathbb{R} \to M$ be the flow constructed in the proof of Proposition 6.1. Since $h \in \mathcal{S}_{\mathrm{id}}(f)$, there exists an isotopy $G : M \times I \to M$ of id_M to h in $\mathcal{S}(f)$. Then is it easy to show that each G_t preserves orbits of $\mathbf{H}$ on some neighbourhood of X, see [10, Lm. 3.4]. Now it follows from [9, Th. 25], see also [13], that there exists a continuous function $\Lambda : (M \setminus \Sigma_f) \times I \to \mathbb{R}$ such that Λ_t is C^∞ for each $t \in I$, $\Lambda_0 = 0$, and $G_t(x) = \mathbf{H}(x, \Lambda_t(x))$ for all $x \in M \setminus \Sigma_f$. Let $\mu : M \to [0, 1]$ be a C^∞ function constant along orbits of H, $\mu = 0$ on X, and $\mu = 1$ on some neighbourhood of $\overline{M \setminus U}$. Define the following map $a : M \times I \to M$ by

$$a(x, t) = \begin{cases} \mathbf{H}(x, \mu(x)\Lambda(x, t)), & x \in U \\ G_t(x), & x \in M \setminus U. \end{cases}$$

We claim that a is an isotopy between id_M and h in $\mathcal{S}(f)$ fixed on some neighbourhood of X.

Since $\mu = 1$ on some neighbourhood of $\overline{M \setminus U}$, we see that a is continuous and a_t is C^∞ for each t. Moreover,

$$a(x,0) = \begin{cases} \mathbf{H}(x,0) = x, & x \in U \\ G_0(x) = x, & x \in M \setminus U. \end{cases}$$

Since h is fixed on U, it follows that $\Lambda(x,1) = 0$ on U. Therefore $\mu\Lambda_1 = \Lambda_1$ and $a_1 = h$. As $\mu = 0$ on X, we obtain that a_t, $(t \in I)$, is fixed on X. □

7. Proof of Theorem 1.7

Suppose $\chi(M) < 0$ and that $f : M \to P$ satisfies (Bd), (Isol), and (SA). We have to find a compact subsurface $X \subset M$ satisfying conditions (1)-(3) of Theorem 1.7.

Construction of X. Let $K_1, \ldots, K_k$ be all the critical components of level-sets of f whose canonical neighbourhoods N_{K_i} have negative Euler characteristic: $\chi(N_{K_i}) < 0$. Since $\chi(M) < 0$, we have by Lemma 3.4 that this collection is non-empty. Denote

$$\mathcal{K} = \mathop{\cup}_{i=1}^{k} K_i,$$

For each $i = 1, \ldots, k$ choose an atom R_i for K_i in a way described in §3.2, and let N_i be the corresponding canonical neighbourhood of K_i. Then we can assume that conditions (i) and (ii) of Lemma 3.3 hold. In particular, $R_i \cap R_j = N_i \cap N_j = \varnothing$ for $i \neq j$.

Denote $\mathcal{R}_{<0} := \mathop{\cup}_{i=1}^{k} R_i$. Let also $B_1, \ldots, B_q$ be all the connected components of $\overline{M \setminus \mathcal{R}_{<0}}$ such that every B_i is a cylinder and f has no critical points in B_i. Put

$$X = \mathcal{R}_{<0} \cup B_1 \cup \cdots \cup B_q.$$

We will show that X satisfies the statement of Theorem 1.7.

Example 7.1. Let M be an orientable surface of genus 2 embedded in $\mathbb{R}^3$ in a way shown in Figure 7.1a) and $f : M \to \mathbb{R}$ be the projection to the vertical line. Critical components of level-sets of f whose canonical neighbourhoods have negative Euler characteristic are denoted by K_1 and K_2. The corresponding surface X is shown in Figure 7.1b).

Before proving Theorem 1.7 we establish the following statement.

Claim 7.2. (i) *Let $h \in \mathcal{S}'(f)$. Then h preserves every leaf $\omega \subset \mathcal{R}_{<0}$ of Δ_f and its orientation.*

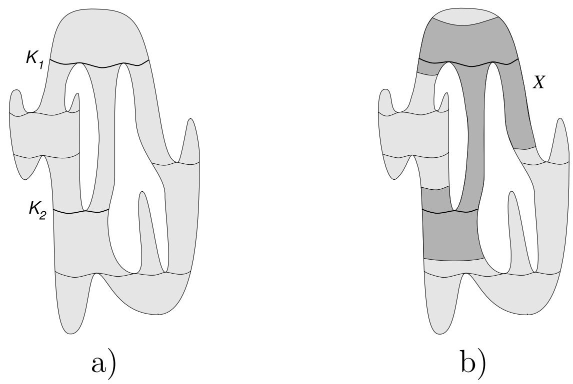

FIGURE 7.1.

(ii) *Suppose h is fixed on a neighbourhood of $\mathcal{R}_{<0}$. Then for every connected component B of $\overline{M \setminus \mathcal{R}_{<0}}$ the restriction $h|_B$ is isotopic to* id_B *with respect to a neighbourhood of $\partial B \cap \mathcal{R}_{<0}$.*

Proof. (i). It follows from the definition of $\mathcal{K}$ that $h(\mathcal{K}) = \mathcal{K}$. We claim that in fact $h(K_i) = K_i$ for all $i = 1, \ldots, k$.

Indeed, suppose that $h(K_i) = K_j$ for some j. Then by Lemma 3.3 $h(R_i) = R_j$ and $h(N_i) = N_j$. On the other hand, since N_i is incompressible, $\chi(N_i) < 0$, and h is isotopic to id_M, it follows from (1) of Proposition 4.5 that $h(N_i) \cap N_i \neq \varnothing$. But $N_i \cap N_j = \varnothing$ for $i \neq j$. Hence $h(N_i) = N_i$ for each $i = 1, \ldots, k$.

Denote by Ξ_i the corresponding partition of N_i, see §5. Since h preserves the set of critical points of f, it follows that h preserves the set of vertexes of degree 2 of K_i. This implies that the restriction of h to N_i yields a certain automorphism h^* of the partition Ξ_i. As $\chi(N_i) < 0$ and h is isotopic to id_M, we get from Theorem 5.3 that h yields a trivial automorphism of Ξ_i. In particular, each (critical) leaf ω of Δ_f in K_i is h^+-invariant.

Let $\omega \subset R_i$ be a regular leaf of Δ_f and $e \subset N_i$ be the corresponding element of Ξ_i containing ω, so e is either an open 2-disk or a half-open cylinder $S^1 \times (0, 1]$. Then

$$\omega \quad = \quad e \quad \cap \quad f^{-1} \circ f(\omega).$$

Notice that $h(e) = e$, since h is Ξ_i-trivial. Moreover, $f \circ h = f$ implies that $h \circ f^{-1} \circ f(\omega) = f^{-1} \circ f(\omega)$, whence $h(\omega) = \omega$. It remains to note that h preserves orientation of ω since it preserves orientation of leaves in K_i.

(ii) Let B be a connected component of $\overline{M \setminus \mathcal{R}_{<0}}$. Then it follows from Corollary 3.8 that B is either

(a) a 2-disk, or
(b) a Möbius band, or
(c) a cylinder such that one of its boundary components belongs to $\mathcal{R}_{<0}$ and another one to ∂M, or
(d) a cylinder with $\partial B \subset \mathcal{R}_{<0}$.

If B is of type (a)-(c), then it is well-known that h is isotopic to id_B with respect to a neighbourhood of $\partial B \cap \mathcal{R}_{<0}$. See [1, 16] for the 2-disk, and [6] for the cases (b) and (c).

Let Q be the union of $\mathcal{R}_{<0}$ with all the components of types (a)-(c). Then we can assume that h is fixed on Q.

It also follows that Q is incompressible and every connected component Q' of Q contains some N_j. This implies that $\chi(Q') \leq \chi(N_j) < 0$. Then by (D) of Proposition 4.5 h is homotopic to id_M via a homotopy fixed on Q. In particular, the restriction of h to every connected component B of type (d) is homotopic in B to id_B relatively ∂B. □

Now we can complete Theorem 1.7.

(1) It follows from the definition of $\mathcal{R}_{<0}$ that ∂X consists of some regular leaves of Δ_f, whence f is locally constant of ∂X. Moreover by Corollary 3.8 every connected component B of $\overline{M \setminus \mathcal{R}_{<0}}$ and therefore of $\overline{M \setminus X}$ is either a 2-disk, or a cylinder, or a Möbius band.

It is also easy to see that B contains critical points of f. Indeed, suppose B is either a 2-disk or a Möbius band. Since f is constant on ∂B, it follows that $f|_B$ is null-homotopic. Hence f must have local extremes in $\mathrm{Int}B$.

On the other hand, if B is a cylinder containing no critical points of f, then by the construction of X we should have that $B \subset X$ which is impossible.

Statement (2) is a particular case of (ii) of Claim 7.2.

(3) We have to show that the inclusion $i : \mathcal{S}'(f, X) \subset \mathcal{S}'(f)$ yields a bijection $i_0 : \pi_0\mathcal{S}'(f, X) \approx \pi_0\mathcal{S}'(f)$.

Claim 7.3. *The map $i_0 : \pi_0\mathcal{S}'(f, X) \to \pi_0\mathcal{S}'(f)$ is an epimorphism.*

Proof. Let $h \in \mathcal{S}'(f)$. We have to show that h is isotopic in $\mathcal{S}'(f)$ to a diffeomorphism fixed on X.

By (i) of Claim 7.2 h preserves the foliation of Δ_f on $\mathcal{R}_{<0}$. Hence by Proposition 6.1 applied to each critical component K_i, $(i = 1, \ldots, k)$, h is isotopic in $\mathcal{S}'(f)$ to a diffeomorphism fixed on some neighbourhood of $\mathcal{R}_{<0}$, so we can assume that h itself is fixed near $\mathcal{R}_{<0}$.

Let B_i, $(i = 1, \ldots, q)$, be a connected component of $\overline{X \setminus \mathcal{R}_{<0}}$. By the construction B_i is a cylinder being a union of regular leaves of Δ_f

and containing no critical points of f. Choose an orientation for B_i. Then we can define a Hamiltonian flow $\mathbf{H} : B_i \times \mathbb{R} \to B_i$ of f on B_i whose orbits are leaves Δ_f belonging to B_i. Notice that h is fixed on some neighbourhood of $\partial B_i \cap \mathcal{R}_{<0}$ and by (ii) the restriction of h to B is homotopic to id_{B_i} relatively ∂B_i. Then by [10, Lm. 4.12] there exists a C^∞ function $\alpha : B_i \to \mathbb{R}$ such that $\alpha = 0$ on some neighbourhood of $\partial B_i \cap \mathcal{R}_{<0}$ and $h(x) = \mathbf{H}(x, \alpha(x))$ for all $x \in B_i$.

Notice that $\partial B_i \cap \mathcal{R}_{<0}$ separates M. Then the map

$$a : M \times I \to M, \qquad a(x,t) = \begin{cases} H(x, t\alpha(x)), & x \in B_i, \\ h(x), & x \in M \setminus B_i \end{cases} \tag{7.1}$$

is an isotopy of h in $\mathcal{S}(f)$ to a diffeomorphism fixed on B_i. Applying this to each B_i we will made h fixed on all of X. □

Claim 7.4. *$i_0 : \pi_0\mathcal{S}'(f,X) \to \pi_0\mathcal{S}'(f)$ is a monomorphism.*

Proof. Let $\mathcal{S}'_{\mathrm{id}}(f)$ and $\mathcal{S}'_{\mathrm{id}}(f,X)$ be the identity path components of $\mathcal{S}'(f)$ and $\mathcal{S}'(f,X)$ respectively. It is clear that $\mathcal{S}'_{\mathrm{id}}(f) = \mathcal{S}_{\mathrm{id}}(f)$. Hence an injectivity of i_0 means that

$$\mathcal{S}'_{\mathrm{id}}(f,X) = \mathcal{S}'(f,X) \cap \mathcal{S}'_{\mathrm{id}}(f) = \mathcal{S}'(f,X) \cap \mathcal{S}_{\mathrm{id}}(f).$$

Evidently, $\mathcal{S}'_{\mathrm{id}}(f,X) \subset \mathcal{S}'(f,X) \cap \mathcal{S}_{\mathrm{id}}(f)$.

Conversely, let $h \in \mathcal{S}'(f,X) \cap \mathcal{S}_{\mathrm{id}}(f)$, so h is fixed on some neighbourhood of X and there exists an isotopy $g_t : M \to M$ in $\mathcal{S}(f)$ between $h_0 = \mathrm{id}_M$ and $h_1 = h$. Then by Proposition 6.3 this isotopy can be made fixed on some neighbourhood of X. Hence $h \in \mathcal{S}'_{\mathrm{id}}(f,X)$. □

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Topology dept., Institute of Mathematics of NAS of Ukraine, Tereshchenkivska st. 3, Kyiv, 01601 Ukraine

E-mail address: `maks@imath.kiev.ua`